A Brief Survey of the Gap Metric for Stability Analysis

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Abstract. *This paper gives a brief literature survey of the gap metric for stability analysis. The gap metric measures the distance between systems in terms of dynamic response and it has a substantial importance for the field of robust control. It is known that over the years the gap metric assumed several extensions implying new features and new calculation methods, but still lacks a timely review. In this contribution we summarize classical results and recent developments following a chronological line. It is observed that although only a few gap metrics are used in real control problems till the present day, this mathematical tool has space for improvement and applications in a more diverse class of linear and nonlinear systems.*

Keywords: *gap metric, robust control, linear and nonlinear systems, functional analysis*

1. INTRODUCTION

In control system problems there is an intrinsic need for measuring the distance between systems. A distance metric can be translated into how well one system approximates another, the sensitivity of the interconnected system to a change of it is components or the uncertainties that can be tolerated in a system without destroying stability when feedback is applied; and the gap metric presents all these features. The gap metric measures the distance between two (possibly unstable) systems in terms of dynamic response and exhibits robustness properties which means that the second system can be viewed as a perturbation or uncertainty regarding the first system with the gap between them indicating if the variation from one system to another preserves closed loop stability. In general the range of values assumed by the gap is [0,1], 0 meaning that the systems are close and can be easily controlled by the same controller, or unity feedback, and 1 meaning that the systems are distant from each other and are difficult to be simultaneously stabilized. Therefore, given a controller design, the gap metric provides a stability guarantee to perturbed systems subject to that same controller.

The stability guarantee is the most important feature of the gap metric concerning applications in real control problems. Systems where stability is a must, such as cooperative systems and rehabilitation robotics can benefit greatly from this aspect in terms of reliability and safety. Also, because the gap metric is able to quantify admissible uncertainties, systems with high nonlinearities, which is the case of chemical process, and systems with several operation points, which is the case of the aerospace industry, can effectively use this mathematical tool for uncertainty analysis and control design. The usual techniques that explores the potential of the gap metric approach in the linear case are multi-model control "Hariprasad et al. (2012)", decentralized control "Lee et al. (2000)", system identification "Geng et al. (2014)" which considers the gap metric as a generalization of the their inequality, and controller certification "Parker et al. (2008)". All of them are based in the discretization of continuous systems which comes from a linear gap metric analysis in the frequency response domain. For the nonlinear gap metric case, the similarity search between nonlinear systems is done in the time domain but still applying the stability test to the state space configuration "Du et al. (2009)".

There are several gap metrics which have been developed serving the purpose of measuring the distance between systems, establishing admissible uncertainties and helping in the design of robust controllers. Each gap metric having specific features, calculation methods and limitations, although they are all topologically equivalents. This equivalence is due to the normed spaces (Lebesgue and Hardy) where the metrics are defined which enables them to work with unstable systems. Although there are several extensions of the gap metric, only two linear gap metrics (Vinnicombe, 1992) and (Georgiou and Smith, 1990) and two nonlinear gap metrics (Georgiou, 1993) and (Vinnicombe, 1998) are usually chosen for the control design of real control problems. This paper gathers more than forty years of intellectual effort and puts it in one framework, besides giving the readers an easy access to the papers concerning the gap metric theory for systems control, in order to provide a general overview of this important mathematical tool.

2. THE GAP TOPOLOGY AND THE STABILITY THEOREM

The gap topology comes from the combination of functional analysis with set theory, and can be regarded as a closed operator from a subspace of the input space to the output space. The pair of input/output signals defines the systems and represents their graph. Frequency response systems and systems that can be put into the state space configuration are closed even if they are unstable (Zames and El-Sakkary, 1981). To date, the gap metrics, can deal only with these types of systems. The definition of the gap metric is stated below.

Definition 2.1 The gap between two closed systems, P_1 and P_2 is defined as the gap between their graphs, $G(P_1)$ and

$G(P_2)$:

$$\delta(P_1, P_2) = \delta(G(P_1), G(P_2)) \tag{1}$$

To attack the problem described by Eq. 1, for linear systems, two main techniques are used, the orthogonal projection in the Lebesgue space (El-Sakkary, 1985) and the coprime factorization in the Hardy space (Quadrat, 2003). Applying these techniques two correspondent general formulas in the frequency domain are obtained:

$$\delta(P_1, P_2) = \left\| \bar{\sigma} \left(\frac{(P_1 - P_2)}{\sqrt{(I + P_1 P_1^*)(I + P_2 P_2^*)}} \right)(s) \right\|_\infty \tag{2}$$

$$\delta(P_1, P_2) = inf_{U \in H^\infty} \left\| \left(\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} U \right)(s) \right\|_\infty \tag{3}$$

where, $\bar{\sigma}$ is the maximum singular value, (*) is the complex conjugate transpose operator, I is the identity matrix, s $\in \mathbb{C}$, and $\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}$ and $\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}$ are coprime representations of P_1 and P_2 respectively.

For nonlinear systems, a pointwise structure "James et al. (2005)" is used to obtained a general formula for the nonlinear gaps in the time domain:

$$\delta(P_1, P_2) = sup_{x_2 \in G(P_2) \cap X} inf_{x_1 \in G(P_1) \cap X} \frac{\|x_2 - x_1\|_p}{\|x_1\|_p}(t) \tag{4}$$

with X being the Lebesgue or Hardy space and p $\in \mathbb{N}$.
The formulas to calculate the linear and nonlinear gap metrics involves finding supremum. This intrinsic characteristic of the gap topology generates conservative results. This conservative behavior can sometimes be dealt with, using loop shaping techniques. Now, the robust stability guarantee using the gap metric approach can be stated.

Theorem 2.1 There is a stabilizing control law u for both P_1 and P_2 if:

$$\delta(P_1, P_2) < \gamma(P_1, u) \tag{5}$$

where $\gamma(P_1, u)$ represents a stability margin or the sensitivity of the standard feedback configuration. The deduction of Eq. 5 can be found in (Vinnicombe, 2001) and (Kato, 2013). It is important to highlight that the gap metric alone does not say much about the robust stability of a given system. By inspecting theorem 2.1 it is observed that along with the gap metric calculation, a stability margin has to established. However, the complete mathematical deduction of the gap metric implies in the existence of stability margins making the inequality presented in theorem 2.1 a mere consequence of the gap metric mathematical structure.

3. EXTENSIONS OF THE GAP METRIC

This section gives a review about the origins of the gap metric alongside the state of the art of this mathematical tool in the control literature, following a chronological line.

3.1 Linear Gap Metrics

The first work published using the gap metric theory for systems control problems in the frequency domain was the PhD thesis of (El-Sakkary, 1982) and then a paper of the same author (El-Sakkary, 1985) summarizing the most import results for the field of control. El-Sakkary initially studied the effects of unstable perturbations in an open loop system and on the same system in feedback, alongside a way to measure the distance (approximation) between two unstable frequency response systems in terms of dynamic response, since the operator norm as a distance measurement works only for stable systems besides being a poor distance measurement for this particular analysis. The gap metric was then used to determine the admissible uncertainties in the open loop configuration which does not impact instability in the closed loop. He used the continuity principle which states that a limited distance between the systems in the open loop configuration implies in a limited distance between them in the closed loop configuration. This fact combined with the admissible uncertainty in the gap metric sense guarantees the existence of at least one stable controller for both systems which is a very important result used in different control problems. For SISO systems El-Sakkary used a spherographic projection or projection into a Riemann sphere to enable the use of the operator norm with unstable systems, and the characteristic projection for MIMO systems. The characteristic projection is an important mathematical tool used especially in quantum mechanics to deal with unbounded operators (Stone, 1951). Although El-Sakkary is metric showed some relevant features, the metric was limited to finite dimension frequency response systems without the presence of RHP poles and zeros simultaneously.

To solve some of the issues of the first gap metric, four gap metrics based on the coprime factorization decomposition were developed. The only way to use the gap metric for the comparison between two systems is by finding previously a homotopy between them which can be represented by the existence of a coprime decomposition. The first coprime factorization based gap metric was introduced by (Vydiasagar, 1984) and is called the graph metric because it comes from the graph topology which is equivalent to the gap topology. This metric, as the following metrics, can calculate the distance between frequency response systems with both open RHP poles and zeros, but due to it is complex mathematical formulation, which involves the field of variational calculus and constrained optimization theory, the gap metric does not possess a direct and analytical method of calculation. As a result, this metric is the least used of all linear gap metrics till the present days. The second gap metric was the first computable gap metric (Georgiou, 1988) and can be calculated by a hankel-toeplitz operator (Georgiou and Smith, 1990). With the second gap metric came the generalized stability margin which is used with all the linear gap metrics to give a robust stability guarantee for the controlled uncertain systems. The third gap metric (Zhu, 1989) is an extension of the first graph metric, where the constrained optimization problem is relaxed in order to facilitate the gap computation. The fourth gap metric, called transpose gap metric (Georgiou and Smith, 1990), uses a different type of coprime factorization from the two metrics described previously and has transpose invariant properties. In the same paper of the transpose gap metric, an optimal generalized stability margin was defined meaning that a maximum admissible uncertainty could be calculated in the gap metric sense. All of these coprime factorization based gap metrics had the same problem of being too conservative because they used their worst case values (overall) as their final values to ensure robust stability. Also, all of them were very time consuming in terms of computational effort.

The frequency wise gap metrics appeared in the literature initially with the work of Vinnicombe in (Vinnicombe, 1991) and (Vinnicombe, 1992), where he used the concept of homotopy of frequency response systems, represented by the winding number condition (WNC), to extend the domain of the systems of the previous gap metrics. This first frequency wise gap metric is called ν-gap metric. The frequency wise response of this metric decreased the conservative results obtained with the coprime factorization based gap metrics. One issue of this metric is that the WNC needed to be satisfied in order to calculate the distance between the systems, and this can be a big problem for systems subjected to certain types of uncertainties. The second frequency wise gap metric, called pointwise gap metric (Qui and Davison, 1992), did not need to satisfy a WNC but on the other hand had more conservative results. The third gap metric, called the abstract ν- gap metric extended the results of the ν-gap and was the first metric to incorporate infinite dimension systems (Ball and Sasane, 2012). The final frequency wise gap metric is considered the generalization of all the linear gap metrics discussed above and can measure the distance between systems with different dimensions. This metric is based on the symmetric gauge functions on the canonical angles of subspaces and it is called the angular metric “Liu et al. (2013)”.

3.2 Nonlinear Gap Metrics

A generalization of the notion of distance between linear systems for nonlinear ones were made for systems in the L^p and H^p spaces in time domain. A new gap metric was initially defined as the differential “diff-gap” metric (Georgiou, 1993) and it was capable of measuring the dynamic distance and to calculate stability margins of some types of nonlinear systems. In this case the robust stability analysis occurs for systems with a differential graph (differentiable manifold) which is a necessary condition for the existence of the diff-gap metric. One important concept treated here was the coordinatization of subspaces which means the decomposition of a metric space into two subspaces to be used as a mean of quantifying stability margins for nonlinear systems and, therefore, guaranteeing robust stability in the gap metric sense.

The second nonlinear gap metric was developed in (Georgiou and Smith, 1995) for nonlinear systems with potential for finite escape over bounded signals in the $L^2[0, \infty]$ domain. The study of uncertainties in nonlinear systems, which included a variety of perturbations in the nominal plant, is presented in (Georgiou and Smith, 1997). Here a limitation was the necessity of a bijection casual map for the calculation of the gap metric.

Attempts to generalize the linear ν-gap metric for nonlinear systems started in (Vinnicombe, 1998) where a method based on Integral Quadratic Contraints (IQCs) is combined with the ν-gap metric topology in order to calculate the distance between nonlinear systems known to satisfy a number of these IQCs. A more general application can be seen in (Vinnicombe, 1999) . Here the systems are defined in the extended Lebesgue space. A direct extension of this metric was developed in the same paper just by changing the domain of the extend Lebesgue space.

Finally a generalization of the linear gap metric for nonlinear systems was obtained in “James et al. (2005)” and compared with the other three nonlinear gap metrics discussed before (excluding the diff-gap metric) in order to achieve a general inference regarding the nonlinear gap structure as whole . One of the most important inferences was that of all these four nonlinear gap metrics needed to satisfy a certain homotopy condition represented by the Hardy and Lebesgue spaces in the time domain, and that there was a single formula to calculate these metrics (Eq. 4). The only difference is the domain of the formula involved in the process of calculation and the p-norm chosen. The choice of these variables depends on the type of the system. Of course, like in the linear gap metrics case, other techniques and new approaches can contribute to create other mathematical formulas although it is still an open problem.

4. CONCLUSION

The gap metric have shown a great potential from the academia and industrial perspective. In this paper, both linear and nonlinear gap metrics are reviewed incorporating classical results along with the state of the art of this metric. Although this paper did not deepen in the mathematical concepts of the gaps or addressed it to real control problems in a more embracing way, the literature presented contains all of the main ideas necessary for a complete comprehension of this subject. All in all, this survey paper helps beginners to get started rapidly in the gap metric theory and in the selection of gaps for future control designs.

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